

The norm of minimal polynomials on several intervals*

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Dedicated to the memory of Franz Peherstorfer

Abstract

Using works of Franz Peherstorfer, we examine how close the n th Chebyshev number for a set E of finitely many intervals can get to the theoretical lower limit $2\text{cap}(E)^n$.

1 Introduction and results

Let $E = \cup_{j=1}^l [a_j, b_j]$, $l > 1$, be a subset of the real line consisting of l disjoint intervals, and let $T_n(x) = x^n + \dots$ be the unique monic polynomial that minimizes the supremum norm $\|T_n\|_E$ among all polynomials of degree n with leading coefficient 1. T_n is called the n -th Chebyshev polynomial of E and its norm $t_n(E) = \|T_n\|_E$ is called the n -th Chebyshev number associated with E . Several authors have investigated Chebyshev polynomials on several intervals, see e.g. [8] by R. M. Robinson and [11] by L. Sodin and P. M. Yuditskii. Franz Peherstorfer also considered them and related quantities in many of his papers (see [4]–[6] and the references therein)—we shall encounter some of his results below.

The present paper is about the behavior of $t_n(E)$. We shall heavily rely on Peherstorfer's findings.

It is an old result of Fekete and Szegő [7, Corollary 5.5.5] that $t_n(E)^{1/n} \rightarrow \text{cap}(E)$, where $\text{cap}(E)$ denotes the logarithmic capacity of E (for the necessary concepts from potential theory see e.g. [7]). It was proved by K. Schiefermayr [12], a student of Franz Peherstorfer, that in all cases we have $t_n(E) \geq 2\text{cap}(E)^n$. Here equality can occur only in very special cases, as is shown by the following proposition, most of which is due to Peherstorfer (see [4, Proposition 1.1]).

Theorem 1 *For a natural number $n \geq 1$ the following are pairwise equivalent.*

a) $t_n(E) = 2\text{cap}(E)^n$.

*AMS Subject Classification 41A10, 31A99. Key words: minimal polynomials, several intervals

[†]Supported by NSF DMS 0700471

- b) T_n has $n + l$ extreme points on E (i.e. $n + l$ points x with the property $|T_n(x)| = \|T_n\|_E$).
- c) $E = \{z \mid T_n(z) \in [-t_n(E), t_n(E)]\}$.
- d) If μ_E denotes the equilibrium measure of E , then each $\mu_E([a_j, b_j])$, $j = 1, 2, \dots, l$ is of the form q_j/n with integer q_j 's ($q_j + 1$ is the number of extreme points on $[a_j, b_j]$).
- e) With $\pi(x) = \prod_{j=1}^l (x - a_j)(x - b_j)$ the equation

$$P_n^2(x) - \pi(x)Q_{n-l}^2(x) = \text{const} > 0$$

is solvable for the polynomials P_n and Q_{n-l} of degree n and $n - l$, respectively.

After Franz Peherstorfer let us call a set E with properties a)—e) for *some* n a T -set. If E is a T -set and n_0 is the minimal degree for which (either of) a)—e) holds, then all other degrees for which a)—e) holds are of the form $n = kn_0$, $k = 1, 2, \dots$ ([4, Proposition 1.1, (i)]). Thus, in this case we have equality in $t_n(E) \geq 2\text{cap}(E)^n$ for infinitely many n . But what about the situation when E is not a T -set, i.e. when $t_n(E) > \text{cap}(E)^n$ for all n ; and in general what can we say about the ratio $t_n(E)/\text{cap}(E)^n$? The following result is due to Widom [15], though it is not stated explicitly in [15].

Theorem 2 *There is a constant C depending only on E such that for all n we have $t_n(E) \leq C\text{cap}(E)^n$, and for infinitely many n we have $t_n(E) \geq (2 + 1/C)\text{cap}(E)^n$.*

Thus, the limit superior of $t_n(E)/2\text{cap}(E)^n$ is always positive and bigger than 1 (this is in sharp contrast with the case of a single interval, when $t_n(E)/2\text{cap}(E)^n$ is identically 1), i.e. for infinitely many n the Chebyshev numbers $t_n(E)$ are bigger by a factor > 1 than the theoretical lower limit $2\text{cap}(E)^n$. However, for infinitely many n they are close to that theoretical lower limit:

Theorem 3 *There is a C such that for infinitely many n we have $t_n(E) \leq (1 + C/n^{1/(l-1)})2\text{cap}(E)^n$.*

This cannot be improved:

Theorem 4 *For every $l > 1$ there are a set E consisting of l intervals and a constant $c > 0$ such that for all n we have $t_n(E) > (1 + c/n^{1/(l-1)})2\text{cap}(E)^n$.*

T -sets, i.e. sets that are inverse images of intervals under a polynomial map, play a distinguished role among sets consisting of finitely many intervals. Indeed, the powerful polynomial-inverse image method is based on them, and a fairly complete theory of orthogonal polynomials can be established on such sets, see e.g. [5] and [6]. It has been proven several times in the literature (see [1], [3], [4], [8] and [14]) that T -sets are dense among all sets consisting

of finitely many intervals. This was extended in [13] to the following: for any set $E = \cup_1^l [a_j, b_j]$ there is a $C > 0$ with the property that for every n there is an $E' = \cup_1^l [a'_j, b'_j]$ such that $|a'_j - a_j|, |b'_j - b_j| \leq C/n$ and $E' = P_n^{-1}[-1, 1]$ with some polynomial P_n of degree n . The argument in Theorems 3, 4 give the following corollary:

Corollary 5 *For any set $E = \cup_1^l [a_j, b_j]$ there is a $C > 0$ with the property that for infinitely many n there is an $E' = \cup_1^l [a'_j, b'_j]$ such that $b'_l = b_l$, $0 \leq b'_j - b_j \leq C/n^{l/(l-1)}$, and $E' = P_n^{-1}[-1, 1]$ with some polynomial P_n of degree n . Furthermore, this is best possible in the sense that there are an $E = \cup_1^l [a_j, b_j]$ and a $c > 0$ such that for all n if $E' = \cup_1^l [a'_j, b'_j]$ is the inverse image of $[-1, 1]$ under a polynomial mapping of degree n , i.e. if $E' = P_n^{-1}[-1, 1]$ with some polynomial P_n of degree n , then $\max_j \{|a_j - a'_j|; |b_j - b'_j|\} \geq c/n^{l/(l-1)}$.*

■

Let us mention that H. Widom [15] gave an asymptotic expression

$$t_n(E) \sim 2\text{cap}(E)^n \nu_n(E)$$

in terms of a variable quantity $\nu_n(E)$ associated with some families of multi-valued analytic functions in $\mathbf{C} \setminus E$. This gives a fairly complete description of $t_n(E)$, however $\nu_n(E)$ is rather implicit and the asymptotics does not seem to be sharp enough to yield e.g. Theorem 3.

2 Proofs

Proof of Theorem 1. That b) implies c) is the implication (i) \Rightarrow (i₂) in [4, Proposition 1.1], and (v) of that proposition shows the equivalence of b) and d) (the measure created in (v) is the equilibrium measure, see e.g. [14, Lemma 2.3]), while (ii) of the same proposition shows the equivalence of b) and e). That c) implies b) is obvious. Thus, it is left to prove the equivalence of a) and c).

If c) holds, then, by [7, Theorem 5.2.5], we have

$$\text{cap}(E)^n = \text{cap}(-\|T_n\|_E, \|T_n\|_E) = \|T_n\|_E/2,$$

which proves a). Finally, to prove that a) implies c) note first that the set

$$E^* = \{x \mid T_n(x) \in [-t_n(E), t_n(E)]\}$$

is always a subset of the real line (see [4, Proposition 1.2]) consisting of finitely many non-degenerate intervals, and clearly $t_n(E) = t_n(E^*)$. Now if c) is not true then E is a proper closed subset of E^* , and hence $\text{cap}(E) < \text{cap}(E^*)$ (note that $\text{cap}(E) = \text{cap}(E^*)$ would mean that the Green's functions with pole at infinity for $\mathbf{C} \setminus E$ and for $\mathbf{C} \setminus E^*$ are the same, which is not the case, since if E

is a proper subset of E^* then $E^* \setminus E$ contains some non-empty interval). But to E^* we can already apply the just proven implication c) \Rightarrow a) to conclude that $t_n(E) = t_n(E^*) = 2\text{cap}(E^*)^n > 2\text{cap}(E)^n$. This shows that if c) is false then so is a), and the proof is over. ■

Incidentally, the very last argument can be used as a proof for the basic inequality $t_n(E) \geq 2\text{cap}(E)^n$ of [12].

Proof of Theorem 3. For (x_1, \dots, x_{l-1}) lying in a small neighborhood U of the origin in \mathbf{R}^{l-1} let

$$E(x_1, \dots, x_{l-1}) = [a_1, b_1 + x_1] \cup [a_2, b_2 + x_2] \cup \dots \cup [a_{l-1}, b_{l-1} + x_{l-1}] \cup [a_l, b_l],$$

and consider

$$\begin{aligned} M(x_1, \dots, x_{l-1}) \\ = (\mu_{E(x_1, \dots, x_{l-1})}([a_1, b_1 + x_1]), \dots, \mu_{E(x_1, \dots, x_{l-1})}([a_{l-1}, b_{l-1} + x_{l-1}])) \end{aligned} \quad (1)$$

where $\mu_{E(x_1, \dots, x_{l-1})}$ denotes the equilibrium measure of the set $E(x_1, \dots, x_{l-1})$. Then $M : U \rightarrow \mathbf{R}^{l-1}$, and it was proved in [14, section 2] (see also Proposition 6 at the end of this paper), that M is a nonsingular C^∞ mapping if U is sufficiently small. In fact, the Jacobian determinant of M is strictly positive and M is 1-to-1 in U .

From the theory of simultaneous Diophantine approximation (see e.g. [2, Theorems VI, VII in Chapter I]) we know that the vector $M(0, \dots, 0)$ can be approximated by a rational vector M_n^* of the form $M_n^* = (p_1/n, \dots, p_{l-1}/n)$ with error $C/n^{l/(l-1)}$:

$$|\mu_E([a_j, b_j]) - p_j/n| \leq C/n^{l/(l-1)} \quad \text{for all } j = 1, \dots, l-1. \quad (2)$$

Then $(v_1, \dots, v_{l-1}) := M^{-1}(M_n^*)$ is of distance $\leq C/n^{l/(l-1)}$ from the origin (with a possibly different C than in (2)), and for these values we get that for $E' = E(v_1, \dots, v_{l-1})$ each of the subintervals $[a_j, b_j + v_j]$ carries a rational portion of the equilibrium measure:

$$\mu_{E'}([a_j, b_j + v_j]) = p_j/n, \quad j = 1, \dots, l-1.$$

Consider now

$$\tilde{E}(x_1, \dots, x_{l-1}, x_l) = [a_1, b_1 + x_1] \cup [a_2, b_2 + x_2] \cup \dots \cup [a_{l-1}, b_{l-1} + x_{l-1}] \cup [a_l, b_l + x_l],$$

and the mapping

$$\begin{aligned} \tilde{M}(x_1, \dots, x_l) \\ = (\mu_{\tilde{E}(x_1, \dots, x_l)}([a_1, b_1 + x_1]), \dots, \mu_{\tilde{E}(x_1, \dots, x_l)}([a_{l-1}, b_{l-1} + x_{l-1}])) \end{aligned} \quad (3)$$

from some $[-a, a]^l$ into \mathbf{R}^{l-1} . This is a C^∞ mapping (see Proposition 6 at the end of this paper), and, as we have just seen, the $(l-1) \times (l-1)$ main minor of its Jacobian

$$\left(\frac{\partial \mu_{\tilde{E}(x_1, \dots, x_l)}([a_i, b_i + x_i])}{\partial x_j} \right)_{i=1, j=1}^{l-1, l}$$

has positive determinant for small a . Furthermore, by the computation given in [14, section 2] the last column of the Jacobian consists of strictly negative entries. Apply now the inverse function theorem (see e.g. [9, Theorem 9.28]) to the equation $\tilde{M}(x_1, \dots, x_l) - M_n^* = 0$. For small x_l the solution is of the form $\tilde{M}(\alpha_1(t), \dots, \alpha_{l-1}(t), t) - M_n^* = 0$, $t \in (-\rho, \rho)$, with some C^∞ functions $\alpha_j = \alpha_{j,n}$ with positive derivative and with $\alpha_j(0) = v_j$, $j = 1, \dots, l-1$ (to be more precise, everything depends on n , but the properties we encounter are uniform in n ; in particular, $\alpha_{j,n}$ have derivative that is bigger than a positive constant independent of n —this follows from the form of the inverse function theorem given in [9, Theorem 9.28]). In other words, for sufficiently small $\rho > 0$ (which is independent of n) and for large enough n there is a one parameter family

$$E'(t) = [a_l, b_l + t] \bigcup_{j=1}^{l-1} [a_j, b_j + \alpha_j(t)], \quad t \in (-\rho, \rho),$$

of sets with the property

$$\mu_{E'(t)}([a_j, b_j + \alpha_j(t)]) = \mu_{E'}([a_j, b_j + v_j]) = p_j/n, \quad j = 1, \dots, l-1,$$

and here the $\alpha_j(t)$'s are C^∞ functions with derivative $\geq \tau > 0$ with some τ independent of n . Furthermore, $|\alpha_j(0)| = |v_j| \leq Cn^{-l/(l-1)}$ for all $1 \leq j \leq l-1$. Therefore, there is a smallest value $\tau_n \geq 0$ of $t \geq 0$ such that $\alpha_j(\tau_n) \geq 0$ for all $1 \leq j \leq l-1$, and then both this τ_n and the values $\alpha_j(\tau_n)$ are at most $C_1/n^{l/(l-1)}$ with some C_1 . Thus, in this case $E \subset E'(\tau_n)$, the left endpoints of the subintervals of E and $E'(\tau_n)$ are the same and the corresponding right endpoints differ by at most $C_1/n^{l/(l-1)}$. According to [13, Lemma 7] this last fact implies $\text{cap}(E'(\tau_n)) \leq (1 + C_2/n^{l/(l-1)})\text{cap}(E)$. Note that on each subinterval of $E'(\tau_n)$ the equilibrium measure has mass of the form p_j/n (this is true for the first $l-1$ subintervals $[a_j, b_j + \alpha_j(\tau_n)]$ by the choice of the $\alpha_j = \alpha_{j,n}$'s and the τ_n 's, and then it is also true for the l -th subinterval $[a_l, b_l + \tau_n]$ since the equilibrium measure has total mass 1). Therefore, according to Theorem 1, we have $t_n(E'(\tau_n)) = 2\text{cap}(E'(\tau_n))^n$, and finally we can conclude

$$\begin{aligned} t_n(E) &\leq t_n(E'(\tau_n)) = 2\text{cap}(E'(\tau_n))^n \leq 2 \left((1 + C_2/n^{l/(l-1)})\text{cap}(E) \right)^n \\ &\leq 2(1 + C_3/n^{1/(l-1)})\text{cap}(E)^n. \end{aligned}$$

■

Proof of Theorem 4. By [2, Theorem III of Chapter V] there are real numbers $\theta_1, \dots, \theta_{l-1}$ and a constant d such that for any n and any integers p_j

we have $\max_j |n\theta_j - p_j| \geq d/n^{1/(l-1)}$. Without loss of generality we may assume $\theta_j > 0$ and $\sum_{j=1}^{l-1} \theta_j < 1$ (just add to θ_j a large number and then divide the result by another sufficiently large number). Now choose a set $E = \cup_{j=1}^l [a_j, b_j]$ such that $\mu_E([a_j, b_j]) = \theta_j$ for $j = 1, \dots, l-1$, and $\mu_E([a_l, b_l]) = 1 - \sum_{j=1}^{l-1} \theta_j$. The existence of such an E follows from [13, Theorem 10]. We claim that this E satisfies the theorem.

Indeed, let n be arbitrary, and consider the Chebyshev polynomial T_n of E . The set

$$E^* = \{x \mid T_n(x) \in [-t_n(E), t_n(E)]\}$$

is a subset of the real line (see [4, Proposition 1.2]) and clearly $t_n(E) = t_n(E^*)$. It was proved by Peherstorfer (see [4, Proposition 1.2]) that this E^* consists of at most $2l-1$ intervals; l of them are “large” intervals $[a_j^*, b_j^*]$ containing one-one subinterval $[a_j, b_j]$, and at most $l-1$ of them are “small” intervals (Peherstorfer called them c -intervals) at most one lying on any (b_j, a_{j+1}) . The equilibrium measure of E^* has a mass of the form (integer/ n) on any component of E^* (see Theorem 1), and it has mass $1/n$ on any c -interval. Therefore, if $\mu_{E^*}([a_j^*, b_j^*]) = p_j/n$, then $n-l+1 \leq \sum_{j=1}^l p_j \leq n$. Since $\mu_E([a_j, b_j]) = \theta_j$, $j = 1, \dots, l-1$, the choice of the θ_j 's gives that for at least one i we have

$$|\mu_{E^*}([a_i^*, b_i^*]) - \mu_E([a_i, b_i])| \geq d/n^{1/(l-1)}. \quad (4)$$

By Proposition 6 at the end of this paper $\mu_E([a_i, b_i])$ is a C^∞ function of the endpoints $\{a_j, b_j\}_{j=1}^l$ of E , hence (4) gives that at least for one $1 \leq j \leq l$ we have either $a_j - a_j^* \geq c_1/n^{1/(l-1)}$ or $b_j^* - b_j \geq c_1/n^{1/(l-1)}$ with some $c_1 > 0$. If we can show that this implies

$$\text{cap}(E^*) \geq (1 + c_2/n^{1/(l-1)})\text{cap}(E), \quad (5)$$

then we shall be ready, for then

$$\begin{aligned} t_n(E) &= t_n(E^*) = 2\text{cap}(E^*)^n \geq 2 \left((1 + c_2/n^{1/(l-1)})\text{cap}(E) \right)^n \\ &\geq 2(1 + c_3/n^{1/(l-1)})\text{cap}(E)^n. \end{aligned}$$

Thus, it is left to prove (5). We may assume e.g. that $b_1^* - b_1 \geq c_1/n^{1/(l-1)}$, and (5) certainly follows if we show that for the sets $\tilde{E}_\delta = [a_1, b_1 + \delta] \cup \cup_{j=2}^l [a_j, b_j]$ we have $\text{cap}(\tilde{E}_\delta) \geq (1 + c\delta)\text{cap}(E)$ with some positive c (and small $\delta > 0$). To this effect note that the equilibrium measure μ_E is the balayage of $\mu_{\tilde{E}_\delta}$ onto E , and in taking this balayage the logarithmic potential

$$U^\mu(z) = \int \log \frac{1}{|z-t|} d\mu(t)$$

changes according to the formula (see e.g. [10, Theorem II.4.4])

$$U^{\mu_E}(z) = U^{\mu_{\tilde{E}_\delta}}(z) + \int_{[b_1, b_1 + \delta]} g_{\mathbf{C} \setminus E}(t, \infty) d\mu_{\tilde{E}_\delta}(t), \quad z \in E,$$

where $g_{\mathbf{C} \setminus E}(t, \infty)$ is Green's function of $\mathbf{C} \setminus E$ with pole at infinity. Since for $z \in E$ we have ([7, Theorem 3.3.4])

$$U^{\mu_E}(z) = \log \frac{1}{\text{cap}(E)}, \quad U^{\mu_{\tilde{E}_\delta}}(z) = \log \frac{1}{\text{cap}(\tilde{E}_\delta)},$$

all is left to show that

$$\int_{[b_1, b_1 + \delta]} g_{\mathbf{C} \setminus E}(t, \infty) d\mu_{\tilde{E}_\delta}(t) \geq c\delta \quad (6)$$

with some $c > 0$. It follows from the explicit formula for the equilibrium measure μ_E given in [14, (2.4)] that with some $c > 0$

$$\frac{d\mu_{\tilde{E}_\delta}(t)}{dt} \geq \frac{c}{\sqrt{b_1 + \delta - t}}, \quad t \in [b_1, b_1 + \delta] \quad (7)$$

for small $\delta > 0$ (see also the derivation of [14, (2.10)]). On the other hand, for $g_{\mathbf{C} \setminus E}(t, \infty)$ we have

$$g_{\mathbf{C} \setminus E}(t, \infty) \geq c\sqrt{t - b_1}, \quad t \in [b_1, b_1 + \delta]. \quad (8)$$

Indeed, notice that

$$g_{\mathbf{C} \setminus E}(z, \infty) \geq c_\gamma g_{\mathbf{C} \setminus [a_1, b_1]}(z, \infty)$$

on any fixed curve γ lying in $\mathbf{C} \setminus E$ and containing $[a_1, b_1]$ (and no other $[a_j, b_j]$) in its interior, and hence by the maximum principle for harmonic functions we have this inequality for all z in the interior of γ . As a consequence,

$$\begin{aligned} g_{\mathbf{C} \setminus E}(t, \infty) &\geq c_\gamma g_{\mathbf{C} \setminus [a_1, b_1]}(t, \infty) = c_\gamma \log |Z + \sqrt{Z^2 - 1}| \\ &\geq c\sqrt{t - b_1}, \quad Z = 2(t - a_1)/(b_1 - a_1) - 1. \end{aligned}$$

Now (7) and (8) clearly give (6), and the proof is over. ■

Proof of Theorem 2. The first claim is implicit in [15], for an alternative proof see [13, Theorem 1]. The second claim also follows from [15] although that is more difficult to see. In any case, it follows from the arguments in the preceding proof. Indeed, no matter what E is (so long as $l > 1$), there are infinitely many n such that

$$|n\mu_E([a_1, b_1]) - p_1| \geq 1/3$$

for all integers p_1 (consider separately the rational and irrational cases for the number $\mu_E([a_1, b_1])$ and note that in the latter case the fractional part of $n\mu_E([a_1, b_1])$, $n = 1, 2, \dots$ is dense in $[0, 1]$). With this we have now instead of (4) the inequality

$$\left| \mu_{E^*}([a_1^*, b_1^*]) - \mu_E([a_1, b_1]) \right| \geq 1/3n,$$

which in turn implies just as before that

$$\max_j \{a_j - a_j^*, b_j^* - b_j\} \geq c_1/n$$

for infinitely many n . The rest of the argument then gives $\text{cap}(E^*) \geq (1 + c_2/n)\text{cap}(E)$, and finally we get as before

$$t_n(E) = t_n(E^*) = 2\text{cap}(E^*)^n \geq 2((1 + c_2/n)\text{cap}(E))^n \geq 2(1 + c_3)\text{cap}(E)^n$$

for infinitely many n . ■

In our considerations we have used several times the following fact, and for completeness we provide a proof for it.

Proposition 6 *If $E = \cup_{j=1}^l [a_j, b_j]$ is a set of disjoint intervals and $\mu_E([a_i, b_i])$ is the mass of the equilibrium measure of E on $[a_i, b_i]$, then $\mu_E([a_i, b_i])$ is a C^∞ function of the a_j, b_j 's.*

Proof. We know (see e.g. [14, Lemma 2.3]) that μ_E is of the form

$$\frac{d\mu_E(t)}{dt} = \frac{|S_{l-1}(t)|}{\pi \prod_{j=1}^l |(t - a_j)(t - b_j)|^{1/2}}, \quad (9)$$

where the coefficients d_k of the polynomial

$$S_{l-1}(t) = t^{l-1} + \sum_{k=0}^{l-2} d_k t^k$$

satisfy the system of equations:

$$\int_{b_i}^{a_{i+1}} \frac{S_{l-1}(t)}{\pi \prod_{j=1}^l |(t - a_j)(t - b_j)|^{1/2}} dt = 0, \quad i = 1, \dots, l-1. \quad (10)$$

This is an inhomogeneous linear system for the d_k 's with matrix

$$\left(\int_{b_i}^{a_{i+1}} \frac{t^k}{\pi \prod_{j=1}^l |(t - a_j)(t - b_j)|^{1/2}} dt \right)_{i=1, k=0}^{l-1, l-2}. \quad (11)$$

If this was singular, then some linear combination of the columns was zero, which would mean that a certain nonzero polynomial of degree at most $l-2$ would have a zero integral (and hence a zero) on each of the $l-1$ intervals (b_i, a_{i+1}) , $i = 1, \dots, l-1$, which is impossible. Hence, the matrix of the system is nonzero. It is well known (and immediate from (10)) that S_{l-1} has precisely one-one zero on every (b_i, a_{i+1}) , $i = 1, \dots, l-1$.

Fix now some points $D_i \in (b_i, a_{i+1})$ and consider instead of the integrals in (11) the integrals

$$\int_{b_i}^{D_i} \frac{t^k}{\pi \prod_1^l |(t - a_j)(t - b_j)|^{1/2}} dt, \quad \int_{D_i}^{a_{i+1}} \frac{t^k}{\pi \prod_1^l |(t - a_j)(t - b_j)|^{1/2}} dt.$$

It follows from Proposition 7 below that all these integrals, and hence all the entries in the above system of equations are C^∞ functions of the a_j, b_j 's, and so, by Cramér's rule, the same is true of the coefficients d_k 's. As an immediate consequence, the zeros of S_{l-1} are also C^∞ functions of the a_j, b_j 's.

Finally, fix points $D'_i \in (a_i, b_i)$ and write

$$\mu_E([a_i, b_i]) = \int_{a_i}^{D'_i} + \int_{D'_i}^{b_i} \frac{|S_{l-1}(t)|}{\pi \prod_1^l |(t - a_j)(t - b_j)|^{1/2}} dt.$$

Since, as we have just seen, the coefficients/zeros of S_{l-1} are C^∞ functions of the a_j, b_j 's and S_{l-1} has all its zeros outside E , the claimed C^∞ property of $\mu_E([a_i, b_i])$ follows from Proposition 7 below (if we apply it to the two terms on the right separately). ■

Proposition 7 *Let (a, b) be a real interval, $b < B$ and $f(t, \alpha, x_1, \dots, x_m)$ a C^∞ function on some domain $(a, B) \times (a, B) \times \Omega$, where Ω is a domain in R^m . Then the integral*

$$I(\alpha, x_1, \dots, x_n) := \int_\alpha^b \frac{f(t, \alpha, x_1, \dots, x_m)}{\sqrt{t - \alpha}} dt$$

is a C^∞ function of $(\alpha, x_1, \dots, x_m)$ in $(a, b) \times \Omega$.

Proof. Integrating by parts we obtain

$$\begin{aligned} \int_\alpha^b \frac{f(t, \alpha, x_1, \dots, x_m)}{\sqrt{t - \alpha}} dt &= 2\sqrt{b - \alpha} f(b, \alpha, x_1, \dots, x_m) \\ &\quad - \int_\alpha^b 2\sqrt{t - \alpha} \frac{\partial f(t, \alpha, x_1, \dots, x_m)}{\partial t} dt. \end{aligned}$$

Repeating the same process k times we find that

$$I(\alpha, x_1, \dots, x_n) = C^\infty \text{ term} + C_k \int_\alpha^b (t - \alpha)^{(2k-1)/2} \frac{\partial^k f(t, \alpha, x_1, \dots, x_m)}{\partial t^k} dt.$$

Therefore, by elementary calculus, the derivative of the left hand side with respect to α exists and equals

$$\frac{\partial I(\alpha, x_1, \dots, x_n)}{\partial \alpha} = C^\infty \text{ term} + C_k \int_\alpha^b (t - \alpha)^{(2k-1)/2} \frac{\partial^{k+1} f(t, \alpha, x_1, \dots, x_m)}{\partial t^k \partial \alpha} dt$$

$$\begin{aligned}
& - C_k \int_{\alpha}^b \frac{2k-1}{2} (t-\alpha)^{(2k-3)/2} \frac{\partial^k f(t, \alpha, x_1, \dots, x_m)}{\partial t^k} dt \\
& - C_k (t-\alpha)^{(2k-1)/2} \frac{\partial^k f(t, \alpha, x_1, \dots, x_m)}{\partial t^k} \Big|_{t=\alpha},
\end{aligned}$$

and here the last term vanishes. Repeating the process we obtain that for $r < k$

$$\frac{\partial^r I(\alpha, x_1, \dots, x_m)}{\partial \alpha^r}$$

is a linear combination of a C^∞ function and of the integrals

$$\int_{\alpha}^b (t-\alpha)^{(2k-2s-1)/2} \frac{\partial^{k+r-s} f(t, \alpha, x_1, \dots, x_m)}{\partial t^k \partial \alpha^{r-s}} dt$$

with $0 \leq s \leq r$. Finally, this shows that for any β_1, \dots, β_m

$$\frac{\partial^{r+\beta_1+\dots+\beta_m} I(\alpha, x_1, \dots, x_m)}{\partial \alpha^r \partial x_1^{\beta_1} \dots \partial x_m^{\beta_m}} \quad (12)$$

exists and is a linear combination of a C^∞ function and of the integrals

$$\int_{\alpha}^b (t-\alpha)^{(2k-2s-1)/2} \frac{\partial^{k+r-s+\beta_1+\dots+\beta_m} f(t, \alpha, x_1, \dots, x_m)}{\partial t^k \partial \alpha^{r-s} \partial x_1^{\beta_1} \dots \partial x_m^{\beta_m}} dt, \quad 0 \leq s \leq r.$$

Again, by elementary calculus, all these are continuous and we can conclude the existence and continuity of (12). Since here $r, \beta_1, \dots, \beta_m$ are arbitrary, the proof is over. ■

Acknowledgement. The author is grateful for the referees for calling his attention to the papers [8] and [11] dealing with Chebyshev polynomials on several intervals.

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